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Progressive radial cross-waves

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Progressive radial cross-waves in a deep, periphractic wavetank are investigated on the assumption that the vertical component of the capillary force vanishes at the wavemaker. For a cylindrical wavemaker, the envelope of the radial cross-wave is shown to obey an evolution equation that differs from the cubic Schrödinger equation only in the presence of a factor $1/R$ in the cubic term, where R is a slow radial variable. Weak, linear damping is incorporated, and the transition conditions at which the directly forced concentric wave loses stability to a parametrically forced cross-wave are obtained. The cylindrical problem is used to develop an asymptotic approximation to the corresponding problem for a spherical wavemaker. The theory is compared with the experiments of Tatsuno, Inoue & Okabe (1969). The theoretical predictions of resonant wavenumbers are consistent with their data, but the corresponding predictions of wavemaker amplitudes, on the assumption of linear damping that is confined to an inextensible (fully contaminated) free-surface boundary layer, are an order of magnitude smaller than those observed by Tatsuno *et al.* (1969). This underprediction of the transition amplitudes may be due to nonlinear phenomena – in particular, nonlinear effects at the contact line and 'undersurface flows' (Taneda 1991) – that are not comprehended by the theoretical model.

1. Introduction

Radial cross-waves were discovered by Faraday in 1831 (see Martin 1932). He subjected a partially immersed cork to a vertical vibration and observed that 'So soon as the cork touched the water a beautiful store of [radial] ridges formed all round it, running out 2, 3 or even 4 inches'. They have since been studied experimentally by Schuler (1933), Tatsuno, Inoue & Okabe (1969, hereinafter referred to as TIO), and Taneda (1991), who observed the transition from outwardly propagating concentric waves to radially decaying cross-waves and confirmed Faraday's observation (in a related context) that, whereas the concentric waves have the same frequency as the wavemaker, the cross-waves have half that frequency. They have been treated analytically by Becker & Miles (1991, hereinafter referred to as I), who considered standing waves in a circular annulus. They are closely related to cross-waves in a rectangular tank, which also were discovered by Faraday and have been reviewed by Miles & Henderson (1990).

TIO measured the transition (from concentric waves to cross-waves or *vice versa*) amplitudes a (in the present notation) of the vertical oscillation of spheres of four different diameters ($2r_1 = 3, 4, 5, 6$ cm) versus the transition frequencies of the cross-waves. They then calculated the corresponding wavenumbers k from the linear dispersion relation on the assumption that the surface tension was 74 dynes/cm and found that their data for the concentric $\rightarrow$ radial and radial $\rightarrow$ concentric transitions collapsed onto two distinct curves, along which a/r_1 falls from 0.3 to 0.005 as kr_1 increases from 2 to 32. They also found that the azimuthal wavenumber m of the radial cross-wave is approximated by $m \sim 1.2kr_1$ over the range of their data.

The direct formulation of the radial cross-wave problem for a spherical wavemaker does not permit separation of variables; accordingly, we first consider a cylindrical wavemaker with the prescribed, radial displacement

$$r = r_1 + \chi(z,t), \quad \chi = af(kz)\sin 2\omega t \quad (0 < \psi < 2\pi), \tag{1.1a, b}$$

where (r, ψ, z) are cylindrical polar coordinates (ψ replaces θ in I §2), on the assumptions (in addition to that of irrotational motion) that

$$ka \equiv \epsilon \ll 1, \quad kd \gg 1, \quad k^2T/g \equiv k^2 l_c^2 \equiv \lambda = O(1), \tag{1.2a–c}$$

$$\omega^2 - \omega_k^2 = O(\epsilon^2\omega^2), \quad |\lambda - \tfrac{1}{2}| \gg \epsilon^2, \tag{1.2d, e}$$

where k is the wavenumber of the radial cross-wave, d is the fluid depth, T is the kinematic surface tension, $l_c \equiv (T/g)^{\frac{1}{2}}$ is the capillary length, and

$$\omega_k^2 \equiv gk + Tk^3 = gk(1+\lambda). \tag{1.3}$$

The assumption (1.2e) rules out a Wilton-type resonance between the cross-wave and its second harmonic. We then use the solution of this cylindrical problem for $kr_1 \gg 1$ to obtain an asymptotic approximation to the solution of the corresponding problem for a spherical wavemaker of radius r_1, the centre of which executes the vertical oscillation $z = -a\sin 2\omega t$.

The linearized, axisymmetric response of the liquid in the periphractic domain bounded by the cylindrical wavemaker (1.1) and a free surface $z = \zeta$ is determined by an extension of Havelock's (1929) gravity-wave solution to capillary–gravity waves on the (conventional) assumption that the vertical component of the capillary force vanishes at the contact line.† This solution is stable for sufficiently small a and ω, but, as either a is increased with ω fixed or conversely, it loses stability to a radial cross-wave. We pose the cross-wave in the form

$$\zeta = (2\epsilon)^{\frac{1}{2}}k^{-1}\,\mathrm{Re}\,[A(R,\tau)\,\mathrm{e}^{-\mathrm{i}\omega t}]\left\{\frac{J_m(kr)\,Y_m'(kr_1) - J_m'(kr_1)\,Y_m(kr)}{[J_m'^2(kr_1) + Y_m'^2(kr_1)]^{\frac{1}{2}}}\right\}\cos m\psi, \tag{1.4}$$

where A is a dimensionless, slowly varying, complex amplitude, $R = 2\epsilon kr$ and $\tau = \epsilon^2\omega t$ (cf. I, where $\tau = \epsilon\omega t$) are slow variables, Re denotes the *real part of*, J_m and Y_m are Bessel functions of order m, and the primes signify differentiation with respect to the argument.

Progressive radial cross-waves resemble progressive cross-waves in a rectangular wavetank, but there are significant differences. For a rectangular wavetank, the linear approximation to the progressive cross-wave is given by (cf. Miles & Becker 1988)

$$\zeta_1 = \sqrt{2\epsilon}\,k^{-1}\,\mathrm{Re}\,[A(X,\tau)\,\mathrm{e}^{-\mathrm{i}\omega t}]\cos k_m y \quad (k_m \equiv m\pi/b), \tag{1.5}$$

where $X = 2\epsilon kx, \tau = \epsilon^2\omega t$, b is the channel breadth, x is the down-channel coordinate and y is the cross-channel coordinate. Comparing (1.4) and (1.5) with (r, R, ψ) and $(x,$

† The axisymmetric velocity potential for the spherical wavemaker in the asymptotic limit $kr_1 \uparrow \infty$ is independent of the contact line condition (see §7), but this may not hold for the cross-wave. The solution for the cylindrical wavemaker with a prescribed wave slope (zero in the present case) at the contact line is given by Rhodes-Robinson (1971b).

$X, \pi y/b$) as ordered counterparts, we observe that (1.5) is independent of x, whereas (1.4) depends on both r and R. Both (1.4) and (1.5) imply no net energy flux across surfaces parallel to the wavemaker, neither of them satisfies a radiation condition (there is no source of energy), and both have infinite energy in this first approximation (in which X, R and τ are fixed). This last difficulty is resolved in the second approximation, in which A decays exponentially as R or $X \uparrow \infty$ (see §6). We further remark that (1.5) is a cutoff mode: the x-component of the linear group velocity is zero, which implies that the energy transferred from the wavemaker to the cross-wave through weak nonlinear interactions remains trapped near the wavemaker (cf. Jones 1984). No direct counterpart of the cutoff mode (1.5) exists for the cylindrical wavemaker; however, the radial group velocity vanishes at $r = m/k$ (the caustic circle), whence we anticipate that $k = O(m/r_1)$ in (1.4).

We begin the determination of $A(R, \tau)$ with a variational formulation of the boundary-value problem in §2, following I and extending that calculation to incorporate surface tension. We then, in §3 and Appendices A and B, develop the trial functions for a weakly nonlinear motion that comprises axisymmetric motion at the driving frequency 2ω, the slowly modulated cross-wave (1.4), and the self-interaction of this cross-wave, and calculate the average Lagrangian in §4 and Appendix C. In §5, we obtain the evolution equation and boundary conditions for $A(R, \tau)$ and incorporate weak, linear damping. This evolution equation, which differs from the cubic Schrödinger equation in the presence of a factor R^{-1} in the cubic term, admits a solution $A = 0$ that corresponds to axisymmetric motion and loses stability to steady cross-wave motion at the threshold (see §6)

$$\epsilon = P^{-1}(\gamma\delta)^{\frac{1}{2}}, \quad \omega = \omega_k, \qquad (1.6a, b)$$

where $\frac{1}{2}\gamma$ is the ratio of the group velocity to the phase velocity for a plane wave of frequency ω, δ is the damping ratio for a free wave of frequency ω, and P is a measure of the parametric forcing of the cross-wave. In §7, we develop the aforementioned equivalence between the cylindrical and spherical wavemakers for $kr_1 \gg 1$ and show that $P \sim kr_1$ and that the wavenumber at which a cross-wave of azimuthal wavenumber m is most easily excited is given by

$$kr_1 = m + O(m^{\frac{1}{3}}). \qquad (1.7)$$

We compare the theory with the experiments of TIO in §8 on the assumption that the surface is fully contaminated (as is typical for laboratory experiments in the absence of rather special precautions; TIO do not report damping measurements). The condition (1.7), which implies that cross-wave excitation is most efficient at (or near) that wavenumber for which the caustic circle (or, equivalently, the turning point of Bessel's equation) is at the wavemaker (as in I), is consistent with TIO's data. The comparison of (1.6) with TIO's data is less satisfactory in that the theory underestimates the observed transition amplitudes by an order of magnitude. Plausible reasons for this discrepancy are the presence of nonlinear effects at the contact line and 'undersurface flows' (Taneda 1991) that are not comprehended by the theoretical model (note that $0.16 < \epsilon < 0.82$ for TIO's data, see figure 3).

Our formulation provides the basis for the determination of the cross-wave domain above the threshold, but, without better agreement between our predicted threshold and observation, we rest content with the construction (in §6) of an approximation to A just above the threshold.

2. Variational formulation

Luke's (1967) Lagrangian, augmented by the capillary energy, is given by

$$L = -\iiint [\phi_t + \tfrac{1}{2}(\nabla\phi)^2 + gz]\,\mathrm{d}V - T\iint (\Gamma - 1)\,\mathrm{d}S, \tag{2.1}$$

where ϕ is the velocity potential, ζ is the free-surface displacement,

$$\Gamma \equiv [1 + (\nabla\zeta)^2]^{\frac{1}{2}}, \tag{2.2}$$

the first integral is over the periphractic volume bounded by the free surface and the wavemaker, and the second integral is over the free surface. An equivalent form, which follows from (2.1) through Green's theorem, the assumption that the temporal average $\langle r\phi\phi_r\rangle$ vanishes as $r\uparrow\infty$, and partial integration with respect to t, is [cf. I (2.7)]

$$2L = \iiint \phi\nabla^2\phi\,\mathrm{d}V + \iint [\phi(2\zeta_t - \phi_z + \nabla\phi\cdot\nabla\zeta) - g\zeta^2 - 2T(\Gamma - 1)]_{z=\zeta}\,\mathrm{d}S$$

$$+ \iint [\phi(\phi_r - \nabla\chi\cdot\nabla\phi - 2\chi_t) - gz^2\chi_z]_{r=r_1+\chi}\,\mathrm{d}W, \tag{2.3}$$

where the three integrals are over the volume, the free surface, and the wavemaker.

The governing equations for ϕ and ζ, obtained by requiring L (2.1) to be stationary with respect to independent variations $\delta\phi$ and $\delta\zeta$ in the cylindrical polar coordinates (r, ψ, z), are

$$\nabla^2\phi = 0 \quad (r_1 + \chi < r < \infty, \quad 0 \leqslant \psi < 2\pi, \quad -\infty < z < \zeta), \tag{2.4}$$

$$\phi_z = \zeta_t + \nabla\phi\cdot\nabla\zeta, \quad \phi_t + \tfrac{1}{2}(\nabla\phi)^2 + g\zeta = T\nabla\cdot(\Gamma^{-1}\nabla\zeta) \quad (z = \zeta), \tag{2.5a, b}$$

$$T\zeta_r = 0, \quad \phi_r = \chi_t + \nabla\phi\cdot\nabla\chi \quad (r = r_1 + \chi), \tag{2.6a, b}$$

together with appropriate radiation and null conditions and the requirement that ϕ and ζ be periodic in ψ. The variation δL differs from that for the gravity-wave problem by

$$-\delta\iint T(\Gamma - 1)\,r\,\mathrm{d}r\,\mathrm{d}\psi = -T\iint \Gamma^{-1}(\nabla\zeta\cdot\delta\nabla\zeta)\,r\,\mathrm{d}r\,\mathrm{d}\psi \tag{2.7a}$$

$$= T\iint \nabla\cdot(\Gamma^{-1}\nabla\zeta)\,\delta\zeta r\,\mathrm{d}r\,\mathrm{d}\psi + T\int \Gamma^{-1}(r\zeta_r\,\delta\zeta)_{r=r_1+\chi}\,\mathrm{d}\psi. \tag{2.7b}$$

The first integral in (2.7 b) contributes to the free-surface condition (2.5 b). The second integral must vanish, which implies either (2.6 a), the natural contact-line condition for the variational principle, or the constraint $\zeta = 0$, which may be more realistic for some configurations (cf. Benjamin & Scott 1979).

3. Trial functions

Proceeding as in I §3, but with provision for a slow radial variation, we pose the trial functions in the dimensionless forms (the scaling anticipates the form of the average Lagrangian)

$$(k^2/\omega)\,\phi = \epsilon^{\frac{1}{2}}\phi_1 + \epsilon(\phi_0 + \phi_{11}) + O(\epsilon^{\frac{3}{2}}), \tag{3.1a}$$

and

$$k\zeta = \epsilon^{\frac{1}{2}}\zeta_1 + \epsilon(\zeta_0 + \zeta_{11}) + O(\epsilon^{\frac{3}{2}}), \tag{3.1b}$$

where: (ϕ_0, ζ_0) represent the axisymmetric response to the wavemaker, (ϕ_1, ζ_1) represent the linear approximation to the cross-wave, and (ϕ_{11}, ζ_{11}) represent the self-interaction of the cross-wave;

$$\phi_0 = \phi_0(\xi, \rho, \theta), \quad \zeta_0 = \zeta_0(\rho, \theta), \quad \phi_1 = \phi_1(\xi, \rho, \psi, \theta; R, \tau), \quad \zeta_1 = \zeta_1(\rho, \psi, \theta; R, \tau), \tag{3.2a–d}$$

and similarly for ϕ_{11} and ζ_{11}, where

$$\xi \equiv kz, \quad \rho \equiv kr, \quad \theta \equiv \omega t, \quad R \equiv 2\epsilon kr, \quad \tau \equiv \epsilon^2 \omega t; \tag{3.3a–e}$$

ϕ_0, ϕ_1 and ϕ_{11} satisfy

$$\phi_{\xi\xi} + \phi_{\rho\rho} + \rho^{-1}\phi_\rho + \rho^{-2}\phi_{\psi\psi} = 0 \quad (-\infty < \xi < 0, \quad kr_1 \equiv \rho_1 < \rho < \infty, \quad 0 \leqslant \psi < 2\pi), \tag{3.4}$$

and appropriate radiation or null conditions as $\rho \uparrow \infty$; ϕ_0, ζ_0, ϕ_1 and ζ_1 satisfy

$$\phi_{n\xi} = \zeta_{n\theta}, \quad \phi_{n\theta} + \Lambda\zeta_n = 0 \quad (\xi = 0), \tag{3.5a, b}$$

$$\lambda\zeta_{n\rho} = 0, \quad \phi_{n\rho} = \delta_{0n}\hat{\chi}_\theta \quad (\rho = \rho_1), \quad \hat{\chi} \equiv f(\xi)\sin 2\theta; \tag{3.6a–c}$$

ϕ_{11} and ζ_{11} satisfy

$$\phi_{11\xi} - \zeta_{11\theta} = -\phi_{1\xi\xi}\zeta_1 + \nabla\phi_1 \cdot \nabla\zeta_1, \quad \phi_{11\theta} + \Lambda\zeta_{11} = -\phi_{1\theta\xi}\zeta_1 - \tfrac{1}{2}(\nabla\phi_1)^2 \quad (\xi = 0), \tag{3.7a, b}$$

$$\lambda\zeta_{11\rho} = 0, \quad \phi_{11\rho} = 0 \quad (\rho = \rho_1); \tag{3.8a, b}$$

δ_{0n} is the Kronecker delta, ∇ now (and subsequently) is the dimensionless (with respect to $1/k$) gradient operator, and

$$\Lambda \equiv (1+\lambda)^{-1}(1-\lambda\Delta), \quad \Delta \equiv \rho^{-1}\partial_\rho\rho\partial_\rho + \rho^{-2}\partial^2_\psi. \tag{3.9a, b}$$

We choose the solution of (3.4)–(3.6*b*) for $n = 1$ (the cross-wave) in the form

$$[\phi_1, \zeta_1] = 2^{\frac{1}{2}}\,\mathrm{Re}\{[-\mathrm{i}e^\xi, 1]\,\boldsymbol{A}(R,\tau)\,\mathrm{e}^{-\mathrm{i}\theta}\}F_m(\rho, \rho_1)\cos m\psi, \tag{3.10}$$

where $\boldsymbol{A}$ is a slowly varying, complex amplitude, m is the azimuthal wavenumber, and

$$F_m(\rho, \rho_1) = \frac{Y'_m(\rho_1)J_m(\rho) - J'_m(\rho_1)Y_m(\rho)}{[Y'^2_m(\rho_1) + J'^2_m(\rho_1)]^{\frac{1}{2}}} \tag{3.11a}$$

$$= \frac{H^{(1)\prime}_m(\rho_1)H^{(2)}_m(\rho) - H^{(2)\prime}_m(\rho_1)H^{(1)}_m(\rho)}{2\mathrm{i}[H^{(1)\prime}_m(\rho_1)H^{(2)\prime}_m(\rho_1)]^{\frac{1}{2}}}. \tag{3.11b}$$

The basic solution of (3.4)–(3.6*b*) obtained by neglecting the slow variation of $\boldsymbol{A}$ with R is a standing wave (cf. I (3.2)) that does not satisfy a radiation condition for $\rho \uparrow \infty$ and has infinite energy

$$\left(\int_{\rho_1}^{\infty} |F_m|^2 \rho \,\mathrm{d}\rho \text{ is divergent}\right).$$

However, allowance for the slow decay of $\boldsymbol{A}$ implies evanescence of the cross-wave at infinity and a finite energy if

$$\int_{R_1}^{\infty} |\boldsymbol{A}|^2 \rho \,\mathrm{d}R$$

converges, and the solution then is physically acceptable (see §6).

The solutions of (3.4)–(3.8) for $[\phi_0, \zeta_0]$ and $[\phi_{11}, \zeta_{11}]$, which are not explicitly required for the present development, are given in Appendices A and B.

4. Average Lagrangian

We now substitute the trial functions (3.1) into the Lagrangian (2.3), expand the integrands in the free-surface and wavemaker integrals about $\xi = 0$ and $\rho = \rho_1$, respectively, separate out L_0, the Lagrangian for axisymmetric motion (which, by definition, is independent of A), and average (indicated by $\langle\ \rangle$) over the fast time θ to obtain the dimensionless average Lagrangian

$$\mathscr{L} \equiv 2(k^5/\omega^2\epsilon^2)\langle L - L_0\rangle = \mathscr{L}_{11} + \mathscr{L}_{011} + \mathscr{L}_{1111} + O(\epsilon). \tag{4.1}$$

$\mathscr{L}_{11}$ is a quadratic functional of the cross-wave that comprises all derivatives (in the Lagrangian) with respect to the slow variables R and τ and is given by (Appendix C)

$$\mathscr{L}_{11} = \mathrm{Re}\int_{R_1}^{\infty}\left[\mathrm{i}\bar{A}A_\tau + \beta\bar{A}A - \left(\frac{1+3\lambda}{1+\lambda}\right)\bar{A}_R A_R\right] w\,\mathrm{d}R, \tag{4.2}$$

where $\bar{A}$ is the complex conjugate of $A, R_1 \equiv 2\epsilon\rho_1$,

$$\beta \equiv (2\epsilon^2\omega^2)^{-1}(\omega^2 - \omega_k^2), \tag{4.3}\dagger$$

and

$$w(\rho) \equiv \pi\rho F_m^2(\rho, \rho_1). \tag{4.4}$$

$\mathscr{L}_{011}$, which is linear in the axisymmetric wave and quadratic in the cross-wave, represents the axisymmetric-cross-wave interaction and is given by (Appendix C)

$$\mathscr{L}_{011} = \mathrm{Re}\,(\mathrm{i}P\bar{A}_1^2), \tag{4.5a}$$

where

$$P = \int_{\rho_1}^{\infty} w(\rho)\,(\{\partial_\xi - \tfrac{1}{2}(1 + \Lambda_\xi^{-1})\,\partial_\xi^2\}\,\Phi_0(\xi,\rho))_{\xi=0}\,\mathrm{d}\rho$$

$$-\tfrac{1}{2}w(\rho_1)\left[\left(1 + \frac{m^2}{\rho_1^2}\right)\int_{-\infty}^{0} f(\xi)\,\mathrm{e}^{2\xi}\,\mathrm{d}\xi - f(0) - 2[\Lambda_\xi^{-1}f(\xi)]_{\xi=0}\right], \tag{4.5b}$$

$$\Lambda_\xi = (1+\lambda)^{-1}(1 + \lambda\,\partial_\xi^2), \tag{4.6}$$

$A_1 \equiv A(R_1, \tau)$, and Φ_0 is the complex amplitude of ϕ_0 (see Appendix A and §7). The operator $\frac{1}{2}(1 + \Lambda_\xi^{-1})\,\partial_\xi^2$ has the limiting forms ∂_ξ^2 for $\lambda \downarrow 0$ and $\frac{1}{2}(1 + \partial_\xi^2)$ for $\lambda \uparrow \infty$ and could be approximated by a bilinear (in λ) interpolation between these limits. $\mathscr{L}_{1111}$ is a quartic functional that represents the self-interaction of the cross-wave and is given by (Appendix C)

$$\mathscr{L}_{1111} = \tfrac{1}{2}Q\int_{R_1}^{\infty} |A|^4\,\frac{\mathrm{d}R}{R} + \tfrac{1}{2}q_{\mathrm{r}}|A_1|^4, \tag{4.7}$$

where the real constants Q and q_{r} are given by (C 14).

We proceed on the hypothesis that (see §6)

$$A = O(\mathrm{e}^{-\kappa R}) \quad (R \uparrow \infty), \quad \kappa = O(1), \quad \mathrm{Re}\,\kappa > 0. \tag{4.8a–c}$$

Substituting (3.11a) into (4.4) and invoking the asymptotic approximations to $J_m(\rho)$ and $Y_m(\rho)$ and $2\rho = R/\epsilon$, we obtain

$$w \sim 1 + \cos\left\{\frac{R}{\epsilon} - (m + \tfrac{1}{2})\,\pi + 2\tan^{-1}\left[\frac{J'_m(\rho_1)}{Y'_m(\rho_1)}\right]\right\} \quad \left(\frac{\rho}{\rho_1} \uparrow \infty\right). \tag{4.9}$$

† Note that β is redefined by (5.6b) in (5.7) *et seq.*

It then follows from Riemann's lemma that (4.2) may be approximated by

$$\mathscr{L}_{11} = \mathrm{Re}\int_{R_1}^{\infty}\left[\mathrm{i}\bar{A}A_\tau + \beta\bar{A}A - \left(\frac{1+3\lambda}{1+\lambda}\right)\bar{A}_R A_R\right]\mathrm{d}R + O(\epsilon). \tag{4.10}$$

Combining (4.4), (4.7) and (4.10) in (4.1), we obtain

$$\mathscr{L} = \int_{R_1}^{\infty}\{\tfrac{1}{2}\mathrm{i}(\bar{A}A_\tau - A\bar{A}_\tau) + \beta\bar{A}A - \gamma A_R\bar{A}_R + \tfrac{1}{2}QR^{-1}A^2\bar{A}^2\}\,\mathrm{d}R$$

$$+\tfrac{1}{2}\mathrm{i}P(\bar{A}_1^2 - A_1^2) + \tfrac{1}{2}q_\mathrm{r}A_1^2\bar{A}_1^2 + O(\epsilon), \tag{4.11}$$

where

$$\gamma \equiv \frac{1+3\lambda}{1+\lambda} \tag{4.12}$$

is twice the ratio of the group velocity to the phase velocity for a wave of frequency ω.

5. Evolution equations

The evolution equations for A and $\bar{A}$ follow from Hamilton's principle in the form

$$\delta\int \mathscr{L}\,\mathrm{d}\tau = 0. \tag{5.1}$$

Substituting (4.11) into (5.1), carrying out the variation with respect to $\bar{A}$, and invoking the null condition at $R=\infty$, we obtain

$$\gamma A_{RR} + \mathrm{i}A_\tau + \beta A + QR^{-1}\bar{A}A^2 = 0 \tag{5.2}$$

and

$$\gamma A_R + \mathrm{i}P\bar{A} + q_\mathrm{r}\bar{A}A^2 = 0 \quad (R=R_1), \quad A\rightarrow 0 \quad (R\uparrow\infty). \tag{5.3a, b}$$

We remark that (5.2) differs from the cubic Schrödinger equation (which governs progressive cross-waves in a rectangular wave tank) only in the presence of the factor R^{-1} in the cubic term. We also remark that, in contrast to cross-waves in a rectangular wave tank, the boundary condition at the wavemaker (5.3a) is nonlinear.

The cross-wave representation (3.10) rests on the assumption of a perfect fluid. We incorporate weak dissipation through the transformation (which follows directly from the necessary form of the evolution equation for any linear oscillator near resonance; cf. I §4)

$$A_\tau \rightarrow A_\tau + \alpha A, \quad \alpha \equiv \delta/\epsilon^2, \tag{5.4a, b}$$

where δ is the damping ratio for the cross-wave (α is the damping ratio on the scale of τ). Linear damping presumably is a consequence of viscous dissipation at the free surface (see §8). Nonlinear viscous damping is negligible in the present approximation, but nonlinear radiation damping associated with the self-interaction solution (ϕ_{11}, ζ_{11}) can be significant, just as for parametrically excited edge waves (cf. Miles 1990), and may be incorporated by replacing q_r by $q \equiv q_\mathrm{r} + \mathrm{i}q_\mathrm{i}$ (see Appendix D) in (5.3a).

We anticipate (see §7) that P is real in the present approximation, although, if it were not, $-\tfrac{1}{2}\arg P$ could be absorbed in $\arg A$ and P replaced by $|P|$. We then may re-scale according to

$$R = (\gamma/P)X, \quad \tau = (\gamma/P^2)T, \quad (\alpha,\beta) = (P^2/\gamma)(\hat{\alpha},\hat{\beta}), \quad (Q,q) = P(\hat{Q},\hat{q}), \tag{5.5a–d}$$

drop the hats on α and β or, equivalently, redefine

$$\alpha \equiv \frac{\gamma\delta}{(\epsilon P)^2}, \quad \beta \equiv \frac{\gamma(\omega^2-\omega_k^2)}{2(\epsilon P)^2\omega^2}, \tag{5.6a, b}$$

and transform (5.2)–(5.4) to

$$A_{XX}+\mathrm{i}A_T+(\beta+\mathrm{i}\alpha)A+\hat{Q}X^{-1}\bar{A}A^2 = 0 \tag{5.7}$$

and

$$A_X+\mathrm{i}\bar{A}+\hat{q}\bar{A}A^2 = 0 \quad (X = X_1), \quad A \to 0 \quad (X \uparrow \infty), \tag{5.8a, b}$$

in which $\hat{Q}$ is real and $\hat{q}$ is complex.

6. Linear eigenvalue problem

Neglecting the cubic terms in (5.7) and (5.8a) and positing

$$A(X,T) = F(X)\,\mathrm{e}^{\sigma T}, \tag{6.1}$$

where σ is real, we obtain the linear eigenvalue problem

$$F_{XX}+[\beta+\mathrm{i}(\alpha+\sigma)]F = 0 \tag{6.2}$$

and

$$F_X+\mathrm{i}\bar{F} = 0 \quad (X = X_1), \quad F \to 0 \quad (X \uparrow \infty). \tag{6.3a, b}$$

The solution of (6.2) and (6.3), normalized to $|F| = 1$ at $X = X_1$, is given by [cf. Lichter & Bernoff 1988; Miles & Becker 1988]

$$F_0 = \exp\left[\mathrm{i}(\tfrac{1}{4}\pi+\tfrac{1}{2}\theta)-\kappa(X-X_1)\right] \tag{6.4}$$

and

$$\sigma_0 = (1-\beta^2)^{\frac{1}{2}}-\alpha, \tag{6.5}$$

where

$$\kappa \equiv \mathrm{e}^{-\mathrm{i}\theta} = 2^{-\frac{1}{2}}[(1-\beta)^{\frac{1}{2}}-\mathrm{i}(1+\beta)^{\frac{1}{2}}], \quad \theta = \tfrac{1}{2}\cos^{-1}(-\beta) \quad (0 < \theta < \tfrac{1}{2}\pi) \tag{6.6a, b}$$

(θ no longer stands for ωt).

It follows from the preceding paragraph that the directly forced axisymmetric motion (for which $F = 0$) is stable for all β if $\alpha > 1$ or for $\beta^2 < 1-\alpha^2$ if $\alpha < 1$ and bifurcates to a parametrically forced cross-wave at

$$\beta = \pm(1-\alpha^2)^{\frac{1}{2}} \quad (\alpha < 1), \quad A = 0. \tag{6.7a, b}$$

The cross-wave threshold is given by

$$\alpha = 1, \quad \beta = 0, \tag{6.8a, b}$$

which, through (5.6), are equivalent to (1.6a, b).

Whether the concentric-cross-wave bifurcation for $\alpha \uparrow 1, \beta \to 0$ and $A \to 0$ is subcritical or supercritical could be determined by a Galerkin or centre-manifold projection based on F_0 (cf. Lichter & Bernoff 1988), but the analysis, which is complicated by the cubic term in the boundary condition (5.8a), is rather involved and does not appear to be worth developing in the absence of better agreement between the present threshold prediction and experiment. Still, it is worth emphasizing that the hysteresis in TIO's data implies a subcritical bifurcation.

The solution of (5.7) and (5.8) for a stationary cross-wave just above threshold may be approximated by (cf. Miles & Becker 1988)

$$A = \mathcal{A}F_0(X), \quad \mathcal{A} = O\left(\frac{\sigma_0}{1-\alpha}\right), \tag{6.9a, b}$$

where A is a complex constant. The cross-wave energy, as calculated from (1.4) and equality between the mean potential and kinetic energies, is given by (after factoring out the fluid density)

$$E_1 = (g+Tk^2)(\epsilon^{\frac{1}{2}}/k)^2 \iint |A|^2 F_m^2 \cos^2 m\psi \, \mathrm{d}S \tag{6.10a}$$

$$= 2^{-\frac{3}{2}}(g+Tk^2)\,k^{-4}(\gamma/P)|A|^2 \quad (\beta^2 \ll 1), \tag{6.10b}$$

where the reduction of (6.10a) to (6.10b) follows that of (4.2)–(4.11). We conclude, as anticipated in §3, that the cross-wave has finite energy.

7. Spherical wavemaker

We now consider a spherical wavemaker of radius r_1, the centre of which executes the vertical oscillation

$$z = -a \sin 2\omega t, \tag{7.1}$$

on the assumption that $kr_1 \equiv \rho_1 \gg 1, \lambda$. The asymptotic approximation to the axisymmetric potential in $\rho, \xi = O(\rho_1)$, obtained by approximating the free-surface condition (cf. (3.5))

$$\Lambda \phi_{0\xi} = -\phi_{0\theta\theta} = 4\phi_0 \tag{7.2}$$

by $\phi_0 = 0$ (cf. Ursell 1954; Rhodes-Robinson 1971a), then is given by

$$\phi_0 \sim \rho_1^3\, \xi(\rho^2+\xi^2)^{-\frac{3}{2}} \cos 2\theta \equiv \Phi_0 \cos 2\theta, \tag{7.3}$$

where, here and throughout this section, $\sim$ implies an asymptotic approximation within an error factor of $1+O(1/\rho, \lambda/\rho_1^2)$. Equating $\phi_{0\rho}$ at $\rho = \rho_1$ to $2f(\xi)\cos 2\theta$ [cf. (3.6b, c)], we obtain

$$f(\xi) \sim -\tfrac{3}{2}\rho_1^4\, \xi(\rho_1^2+\xi^2)^{-\frac{5}{2}} \tag{7.4}$$

for the radial displacement function of the equivalent cylindrical wavemaker. We note that

$$f_{\max} = 0.43 \quad \text{at} \quad \xi/\rho_1 = -\tfrac{1}{2} \quad \text{and} \quad \int_{-\infty}^{0} f \, \mathrm{d}\xi = \tfrac{1}{2}\rho_1,$$

which reflects the equality between the volumetric displacements of the sphere and equivalent cylinder.

Substituting Φ_0 and f from (7.3) and (7.4) into (4.5b), we obtain

$$P \sim \int_{\rho_1}^{\infty} \left(\frac{\rho_1}{\rho}\right)^3 w(\rho)\,\mathrm{d}\rho - \frac{3}{16}\left(1+\frac{m^2}{\rho_1^2}\right)\frac{w(\rho_1)}{\rho_1}. \tag{7.5}$$

The last term is negligible in the present approximation, and the integral may be evaluated analytically (Luke 1962, §11.2(20)) to obtain

$$P \sim \left(\frac{\rho_1^3}{m^2-\frac{1}{4}}\right)\left\{1 - \frac{2}{\pi\rho_1^3}\left[\frac{\rho_1^2 - m^2 + \frac{1}{2}}{J_m'^2(\rho_1)+Y_m'^2(\rho_1)}\right]\right\}. \tag{7.6}$$

The wavenumber k, and hence $\rho_1 = kr_1$, at the threshold is determined by $\omega = \omega_k$ (1.6b), while m is determined by the threshold condition that the dimensionless forcing amplitude ϵ (1.6a) be a minimum or, equivalently, that P be a maximum. TIO's experimental data, the numerical results in I, and the discussion following (1.5) all suggest that this maximum occurs near the turning point, $\rho_1 = m$, of Bessel's

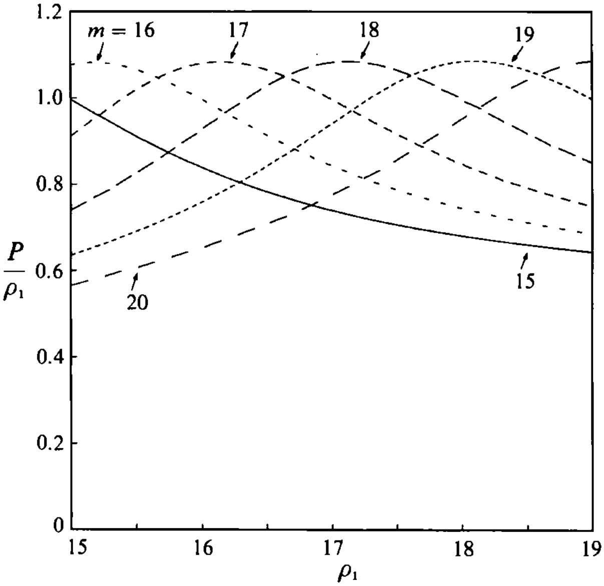

FIGURE 1. P/ρ_1 *vs.* ρ_1 where P/ρ_1 is given by (7.6) and $m = (15, 16, \ldots, 20)$.

equation for $J_m(\rho_1)$ and $Y_m(\rho_1)$, and an asymptotic approximation based on this hypothesis yields

$$m/\rho_1 = 1+O(\rho_1^{-\frac{2}{3}}), \quad (P/\rho_1) = 1+O(\rho_1^{-\frac{1}{3}}). \tag{7.7a, b}$$

Numerical plots of P/ρ_1 *vs.* ρ_1 (7.6) for $m = (15, 16, \ldots, 20)$ are shown in figure 1. The maxima of P/ρ_1 for $m = (4, 8, 20)$ are $(1.01, 1.06, 1.09)$ and occur at $\rho_1 = (3.70, 7.46, 19.07)$.

8. Comparison with experiment

TIO measured the transition amplitudes at which the directly forced, axisymmetric wave lost stability to the radial cross-wave and *vice versa* for spherical wavemakers (submerged to their equators) of four different diameters ($2r_1 = 3, 4, 5, 6$ cm) and forcing frequencies ranging from 10–100 Hz. They also reported the observed azimuthal wavenumber m for these experiments. Using the known forcing frequency, TIO estimated the radial wavenumber k from (1.3) with $T = 74$ dynes/cm and concluded that $m \sim 1.2kr_1$ over the range of their data. Figure 2(*a*) presents kr_1 *vs.* m for TIO's data with $T = 74$ dynes/cm. Fitting these data to a straight line, we find

$$kr_1 = (0.85 \pm 0.01)\, m - (0.02 \pm 0.17), \tag{8.1}$$

in agreement with TIO. However, the water in TIO's experiments does not appear to have been specially treated (it was supplied from the well in the campus of the university), and a more realistic value of surface tension is 50 dynes/cm. Figure 2(*b*) presents kr_1 *vs.* m for TIO's data with $T = 50$ dynes/cm, for which a linear fit yields

$$kr_1 = (0.97 \pm 0.01)\, m - (0.35 \pm 0.19). \tag{8.2}$$

We remark that these values of k are not necessarily resonant wavenumbers, since TIO did not measure resonant frequencies (i.e. they did not determine the frequency, ω_k, at which a cross-wave of fixed azimuthal wavenumber m is most easily excited);

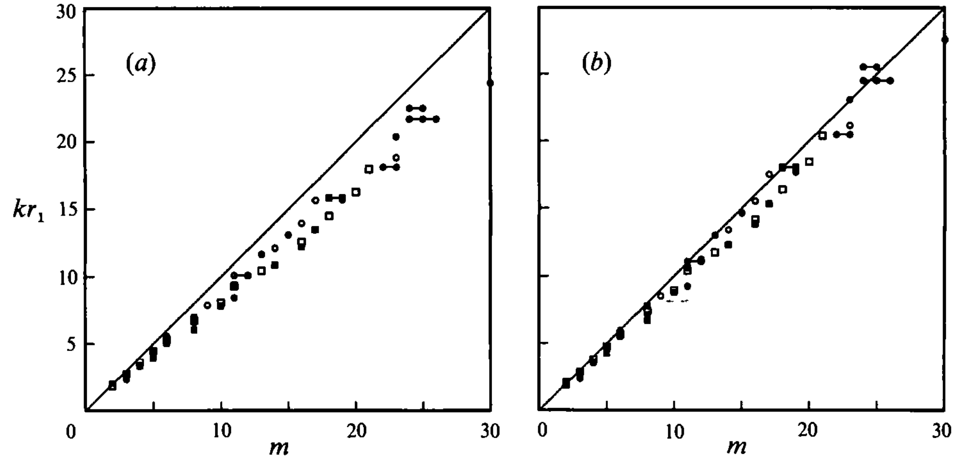

FIGURE 2. kr_1 *vs.* m for the experiments of Tatsuno *et al.* (1969) with (*a*) $T = 74$ dynes/cm (cf. I figure 8*b*) and (*b*) $T = 50$ dynes/cm. Horizontal bars indicate uncertainty in the azimuthal wavenumber m. ■, $r_1 = 1.50$ cm; □, $r_1 = 2.00$ cm; ●, $r_1 = 2.50$ cm; ○, $r_1 = 2.99$ cm.

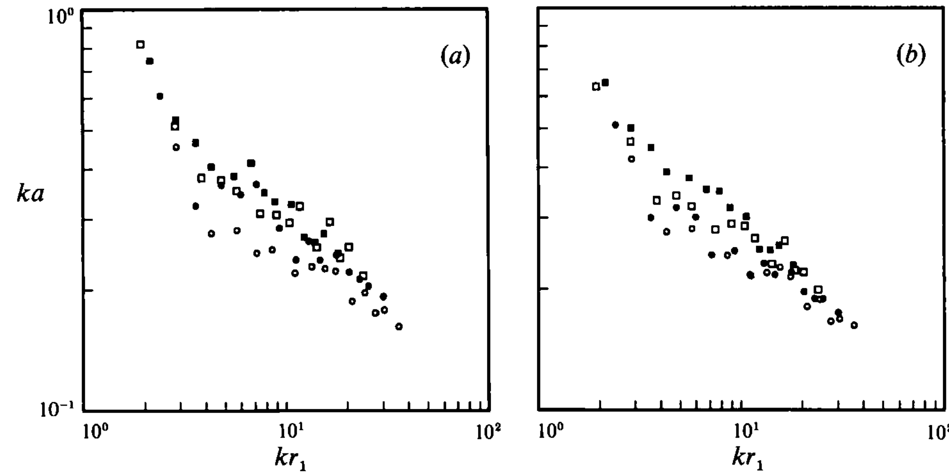

FIGURE 3. ka *vs.* kr_1 for the experiments of Tatsuno *et al.* (1969) with $T = 50$ dynes/cm for (*a*) the concentric to cross-wave transition and (*b*) the cross-wave to concentric transition. ■, $r_1 = 1.50$ cm; □, $r_1 = 2.00$ cm; ●, $r_1 = 2.50$ cm; ○, $r_1 = 2.99$ cm.

accordingly, while no quantitative comparison between theoretical and experimental resonant wavenumbers is possible, either (7.7*a*) or the numerically determined value of ρ_1 at which P (7.6) achieves its maximum with m fixed (see figure 1), is consistent with TIO's data.

Figure 3 presents TIO's measured transition amplitudes, normalized by $1/k$, versus kr_1 on a log–log graph, where k is estimated by (1.3) with $T = 50$ dynes/cm. We find that their data are fitted by

$$ka = C_*(kr_1)^{-\mu}, \tag{8.3}$$

where $\mu = 0.42 \pm 0.03$ and $C_* = 0.76 \pm 0.05$ for the concentric $\rightarrow$ cross-wave transition (figure 3*a*) and $\mu = 0.40 \pm 0.02$ and $C_* = 0.66 \pm 0.04$ for the cross-wave $\rightarrow$ concentric transition (figure 3*b*). For $T = 74$ dynes/cm, the corresponding values of μ are

0.48 ± 0.03 (concentric$\rightarrow$cross-wave transition) and 0.46 ± 0.02 (cross-wave$\rightarrow$concentric transition), while C_* remains essentially unchanged.

The transition amplitude at resonance, obtained by setting $\alpha = 1$ in (5.6a), is given by

$$ka = (\gamma\delta)^{\frac{1}{2}} P^{-1}. \tag{8.4}$$

The damping ratio for a fully contaminated surface is given by (see Miles 1967)

$$\delta = \tfrac{1}{4} k (2\nu/\omega)^{\frac{1}{2}}, \tag{8.5}$$

where ν is the kinematic viscosity. (It follows from dimensional considerations that the viscous damping at the sphere is negligible compared with that at the surface film.)

A representative state reported by TIO has

$$r_1 = 2\ \text{cm}, \quad \omega = 70\pi\ \text{rad/s}, \quad m = 20, \quad \rho_1 = 18.46, \quad a = 0.026\ \text{cm},$$
$$(T = 50\ \text{dynes/cm}).$$

Assuming that an $m = 20$ mode is resonantly excited, using (4.12), (8.5) and (7.6) to obtain $\gamma = 2.6$, $\delta = 0.02$, and $P = 1.09 kr_1$, and invoking (8.4), we obtain $a = 0.001$ cm, which is an order of magnitude smaller than that observed by TIO. For $T = 74$ dynes/cm, the corresponding values of ρ_1/a are 16.30/0.002 cm.

Pursuing the comparison with TIO, we combine (8.4) and (8.5) and invoke $\omega^2 = Tk^3$ and $P \approx kr_1$, to obtain

$$ka \sim C_* (kr_1)^{-\frac{7}{8}}, \quad C_* = 1.03 (Tr_1)^{-\frac{1}{8}} \nu^{\frac{1}{4}}. \tag{8.6a, b}$$

The difference between the exponents -0.4 in (8.3) and $-\frac{7}{8}$ in (8.6a) suggests that the postulated damping mechanism (linear boundary-layer damping beneath an inextensible film) implicit in (8.5) differs significantly from that in TIO's experiments. It may be that phenomena such as nonlinear contact-line damping and 'undersurface flows' (Taneda 1991) that are not comprehended by the present theoretical model are responsible for the discrepancy between the predicted and observed transition amplitudes.

Further progress appears to require more detailed experimental data, including, in particular, the dependence of the transition amplitude for capillary waves on the kinematic surface tension T and the kinematic viscosity ν, perhaps for some fluid (e.g. n-butyl alcohol or filtered and de-ionized water) that gives a more uniform contact line and more reproducible surface conditions than the water used by TIO.

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Appendix A. Cylindrical wavemaker

The solution of (3.4)–(3.6) for axisymmetric gravity waves ($m = \lambda = 0$) is obtained by Havelock (1929) through a Fourier transformation with respect to the vertical coordinate and extended to capillary–gravity waves by Rhodes-Robinson (1971b)

through the development of appropriate Green's functions. We could adapt the latter but instead find it expedient to proceed through a Hankel (Weber) transformation with respect to the radial coordinate.

We introduce complex amplitudes according to

$$[\phi_0, \zeta_0] = \mathrm{Re}\{[\Phi(\xi,\rho), \mathrm{i}Z(\rho)]\,\mathrm{e}^{-2\mathrm{i}\theta}\} \tag{A 1}$$

($\Phi \equiv \Phi_0$ and $Z \equiv Z_0$), the substitution of which into (3.4)–(3.6) yields

$$\nabla^2\Phi = 0, \tag{A 2}$$

$$\Phi_\xi = 2Z, \quad \Lambda Z = 2\Phi \quad (\xi = 0), \tag{A 3a, b}$$

$$\lambda Z_\rho = 0, \quad \Phi_\rho = 2f(\xi) \quad (\rho = \rho_1). \tag{A 4a, b}$$

Φ also must vanish at $\xi = -\infty$ and satisfy a radiation condition at $\rho = \infty$. Introducing the Hankel-transform pair

$$\hat{\Phi}(\xi,\mu) = \int_{\rho_1}^{\infty} \Phi(\xi,\rho)\,F_0(\mu\rho,\mu\rho_1)\,\rho\,\mathrm{d}\rho, \quad \Phi(\xi,\rho) = \int_0^\infty \hat{\Phi}(\xi,\mu)\,F_0(\mu\rho,\mu\rho_1)\,\mu\,\mathrm{d}\mu, \tag{A 5a, b}$$

where F_0 is the Bessel function (3.11), we transform (A 2)–(A 4) to

$$\hat{\Phi}_{\xi\xi} - \mu^2\hat{\Phi} = 2\rho_1 F_0(\mu\rho_1,\mu\rho_1) f(\xi), \tag{A 6}$$

$$\hat{\Phi}_\xi = 2\hat{Z}, \quad \kappa\hat{Z} = 2\hat{\Phi} \quad (\xi = 0), \quad \kappa \equiv \frac{1+\lambda\mu^2}{1+\lambda}. \tag{A 7a–c}$$

The solution of (A 6) and (A 7) is given by

$$\hat{\Phi} = -\frac{\rho_1 F_0(\mu\rho_1,\mu\rho_1)}{\mu}\int_{-\infty}^{0} f(\eta)\left[\mathrm{e}^{-\mu|\xi-\eta|} + \left(\frac{\kappa\mu+4}{\kappa\mu-4}\right)\mathrm{e}^{\mu(\xi+\eta)}\right]\mathrm{d}\eta. \tag{A 8}$$

Substituting (A 8) into (A 5b) and invoking (3.11b) for F_0, we obtain

$$\Phi = (\mathrm{i}\pi)^{-1}\int_{-\infty}^{0} f(\eta)\,\mathrm{d}\eta \int_0^\infty \left[\frac{H_0^{(1)}(\mu\rho)}{H_0^{(1)\prime}(\mu\rho_1)} - \frac{H_0^{(2)}(\mu\rho)}{H_0^{(2)\prime}(\mu\rho_1)}\right]\left[\mathrm{e}^{-\mu|\xi-\eta|} + \left(\frac{\kappa\mu+4}{\kappa\mu-4}\right)\mathrm{e}^{\mu(\xi+\eta)}\right]\frac{\mathrm{d}\mu}{\mu}, \tag{A 9}$$

where the path of integration is indented under the pole at $\kappa\mu = 4$ ($\mu = \mu_0$) in order to satisfy the radiation condition at $\rho = \infty$. The corresponding result for the free-surface displacement, obtained through the substitution of (A 9) into (A 3a), is

$$Z = \frac{4}{\mathrm{i}\pi}\int_{-\infty}^{0} f(\eta)\,\mathrm{d}\eta \int_0^\infty \left[\frac{H_0^{(1)}(\mu\rho)}{H_0^{(1)\prime}(\mu\rho_1)} - \frac{H_0^{(2)}(\mu\rho)}{H_0^{(2)\prime}(\mu\rho_1)}\right]\frac{\mathrm{e}^{\mu\eta}\,\mathrm{d}\mu}{\kappa\mu-4}. \tag{A 10}$$

We separate (A 9) into radiated and trapped waves by deforming the path of integration for the $H_0^{(1)}/H_0^{(2)}$ component to the positive/negative imaginary axis of the complex-μ plane. The $H_0^{(1)}$ component then contributes both radiated (from the pole at $\mu = \mu_0$) and trapped waves, whereas the $H_0^{(2)}$ component contributes only a trapped wave. The end result is

$$\Phi = 4\left(\frac{1+\lambda\mu_0^2}{1+3\lambda\mu_0^2}\right)\exp(\mu_0\xi)\frac{H_0^{(1)}(\mu_0\rho)}{H_0^{(1)\prime}(\mu_0\rho_1)}\int_{-\infty}^{0} f(\eta)\exp(\mu_0\eta)\,\mathrm{d}\eta$$
$$+\frac{4}{\pi}\int_0^\infty \frac{K_0(\nu\rho)}{K_0'(\nu\rho_1)}\sin(\nu\xi+\upsilon)\frac{\mathrm{d}\nu}{\nu}\int_{-\infty}^{0} f(\eta)\sin(\nu\eta+\upsilon)\,\mathrm{d}\eta, \tag{A 11}$$

where μ_0 is the real root of

$$\mu_0(1+\lambda\mu_0^2) = 4(1+\lambda) \tag{A 12}$$

and

$$v = \tan^{-1}\left[\frac{\nu(1-\lambda\nu^2)}{4(1+\lambda)}\right]. \tag{A 13}$$

Setting $\lambda = 0$ in (A 11)–(A 13), we recover Havelock's (1929) result.

Appendix B. The self-interaction problem

We consider the solution of Laplace's equation, (3.4), subject to (3.7) and (3.8). Substituting (3.10) into (3.7), we transform the free-surface conditions to

$$\phi_\xi - \zeta_\theta = \mathrm{Re}\{\mathrm{i}A^2\mathscr{G}\,\mathrm{e}^{-2\mathrm{i}\theta}\}, \quad \phi_\theta + \Lambda\zeta = \tfrac{1}{2}|A|^2\mathscr{G} + \mathrm{Re}\{A^2\mathscr{H}\,\mathrm{e}^{-2\mathrm{i}\theta}\} \quad (\xi = 0), \tag{B 1a, b}$$

where $\phi \equiv \phi_{11}, \zeta \equiv \zeta_{11}, A \equiv A(R,\tau)$,

$$\mathscr{G} \equiv \mathscr{F}_m^2 - (\nabla\mathscr{F}_m)^2, \quad \mathscr{H} \equiv \tfrac{3}{2}\mathscr{F}_m^2 + \tfrac{1}{2}(\nabla\mathscr{F}_m)^2, \tag{B 2a, b}$$

$$\mathscr{F}_m = \mathscr{F}_m(\rho,\psi) \equiv F_m(\rho,\rho_1)\cos m\psi, \tag{B 2c}$$

and $F_m(\rho,\rho_1)$ is given by (3.11). The solution has the form (in which $\langle\zeta\rangle$ is a non-zero, temporal mean displacement)

$$[\phi, \zeta - \langle\zeta\rangle] = \mathrm{Re}\{A^2[\mathrm{i}\Phi(\xi,\rho,\psi), Z(\rho,\psi)]\,\mathrm{e}^{-2\mathrm{i}\theta}\}, \quad \langle\zeta\rangle = \tfrac{1}{2}|A|^2\Lambda^{-1}\mathscr{G}(\rho,\psi), \tag{B 3a, b}$$

the substitution of which into (3.4), (B 1) and (3.8) yields

$$\nabla^2\Phi = 0, \tag{B 4}$$

$$\Phi_\xi + 2Z = \mathscr{G}, \quad 2\Phi + \Lambda Z = \mathscr{H} \quad (\xi = 0), \tag{B 5a, b}$$

$$\lambda Z_\rho = 0, \quad \Phi_\rho = 0 \quad (\rho = \rho_1). \tag{B 6a, b}$$

Proceeding as in Appendix A, we introduce the transform pair (cf. (A 5))

$$\hat{\Phi}_n = \iint \Phi F_n(\mu\rho,\mu\rho_1)\cos n\psi\,\rho\,\mathrm{d}\rho\,\mathrm{d}\psi, \quad \Phi = \sum_n \left(\frac{2-\delta_{0n}}{2\pi}\right)\int_0^\infty \hat{\Phi}_n F_n(\mu\rho,\mu\rho_1)\,\mu\,\mathrm{d}\mu\cos n\psi, \tag{B 7a, b}$$

where the summation is over $n = 0$ and $2m$, to obtain

$$\begin{bmatrix}\hat{\Phi}_n\,\mathrm{e}^{-\mu\xi} \\ \hat{Z}_n\end{bmatrix} = \frac{1}{\kappa\mu - 4}\begin{bmatrix}\kappa & -2 \\ -2 & \mu\end{bmatrix}\begin{bmatrix}\hat{\mathscr{G}}_n \\ \hat{\mathscr{H}}_n\end{bmatrix}, \tag{B 8}$$

where $\kappa = \kappa(\mu)$ is given by (A 7c).

It follows from integration by parts and (3.11a) that

$$\iint (\nabla\mathscr{F}_m)^2 F_n \cos n\psi\,\rho\,\mathrm{d}\rho\,\mathrm{d}\psi = (1-\tfrac{1}{2}\mu^2)\iint \mathscr{F}_m^2 F_n \cos n\psi\,\rho\,\mathrm{d}\rho\,\mathrm{d}\psi \quad (n = 0, 2m), \tag{B 9}$$

by virtue of which the transforms of $\mathscr{G}$ and $\mathscr{H}$ (B 2a, b) are given by

$$[\hat{\mathscr{G}}_n, \hat{\mathscr{H}}_n] = \left(\frac{2\pi}{2-\delta_{0n}}\right)[\tfrac{1}{4}\mu^2, 1-\tfrac{1}{8}\mu^2]\,\mathscr{K}_n(\mu) \quad (n = 0, 2m), \tag{B 10}$$

where

$$\mathscr{K}_n(\mu) = \int_{\rho_1}^\infty F_n(\mu\rho,\mu\rho_1)\,F_m^2(\rho,\rho_1)\,\rho\,\mathrm{d}\rho. \tag{B 11}$$

Substituting (B 10) into (B 8) and invoking (B 7b) and its counterpart for Z, we obtain

$$[\Phi, Z] = \tfrac{1}{4}\sum_n \cos n\psi \int_0^\infty [(\kappa\mu^2+\mu^2-8)\,e^{\mu\xi}, 4\mu-2\mu^2-\tfrac{1}{2}\mu^3]\cdot \mathscr{K}_n(\mu)\,F_n(\mu\rho,\mu\rho_1)\frac{\mu\,d\mu}{\kappa\mu-4}, \tag{B 12}$$

where the path of integration passes under the pole at $\mu = \mu_0$, determined by $\kappa\mu = 4$, in order to satisfy the radiation condition at $\rho = \infty$. The asymptotic approximation for $\rho\uparrow\infty$ is dominated by the contribution of this pole and the neighbourhood of $\mu = 0$ and is given by

$$[\Phi, Z] \sim \tfrac{1}{2}[1, -\tfrac{1}{2}\mu_0]\left(\frac{1+\lambda}{1+3\lambda\mu_0}\right)(\mu_0^2+4\mu_0-8)\left(\frac{\pi\mu_0}{2\rho}\right)^{\frac{1}{2}}\exp[i(\mu_0\rho+\tfrac{3}{4}\pi)]$$
$$\sum_n \left[\frac{H_n^{(2)\prime}(\mu_0\rho_1)}{H_n^{(1)\prime}(\mu_0\rho_1)}\right]^{\frac{1}{2}} \mathscr{K}_n(\mu_0)\,e^{\frac{1}{2}in\pi}\cos n\psi + [\mathscr{F}_m^2, -\tfrac{1}{2}\Lambda^{-1}\mathscr{G}] \quad (\xi = 0, \rho\uparrow\infty). \tag{B 13}$$

Appendix C. Reduction of the average Lagrangian

The substitution of the trial functions (3.1) into the Lagrangian (2.3), followed by the expansion of the integrands in the free-surface and wavemaker integrals about $\xi = 0$ and $\rho = \rho_1$, respectively, averaging over θ, and the subdivision (4.1), yields

$$\begin{aligned}\mathscr{L}_{11} = {} & 2\iiint \langle \phi_1(\phi_{1R}+2\rho\phi_{1\rho R}+2\epsilon\rho\phi_{1RR})\rangle\, d\rho\, d\psi\, d\xi + 2\iint \langle \rho\phi_1\phi_{1R}\rangle_{\rho=\rho_1} d\psi\, d\xi \\ & +\left(\frac{2\lambda}{1+\lambda}\right)\left(\iint \langle \zeta_1(\zeta_{1R}+2\rho\zeta_{1\rho R}+2\epsilon\rho\zeta_{1RR})\rangle\, d\rho\, d\psi + \int \langle \rho\zeta_1\zeta_{1R}\rangle_{\rho=\rho_1} d\psi\right) \\ & +\iint \left\langle \phi_1\zeta_{1\tau} + \frac{1}{2\epsilon^2}\left\{\phi_1\zeta_{1\theta} - \left(\frac{gk}{\omega^2}\right)[\zeta_1^2+\lambda(\nabla\zeta_1)^2]\right\}\right\rangle_{\xi=0} \rho\, dR\, d\psi,\end{aligned} \tag{C 1}$$

$$\begin{aligned}\mathscr{L}_{011} = {} & \iint \langle \phi_{11}\zeta_{0\theta}+\phi_0\zeta_{11\theta}-2(1+\lambda)^{-1}(\zeta_0\zeta_{11}+\lambda\nabla\zeta_0\cdot\nabla\zeta_{11})+\phi_{0\xi}\zeta_1\zeta_{1\theta}+\phi_{1\xi}(\zeta_0\zeta_1)_\theta \\ & +\phi_1[-\phi_{0\xi\xi}\zeta_1-\phi_{1\xi\xi}\zeta_0+\nabla\phi_0\cdot\nabla\zeta_1+\nabla\phi_1\cdot\nabla\zeta_0]\rangle\,\rho\, d\rho\, d\psi \\ & +\iint \langle -\phi_{11}\hat{\chi}_\theta+\phi_1(\phi_{1\rho\rho}\hat{\chi}-\phi_{1\xi}\hat{\chi}_\xi)\rangle\,\rho_1\, d\psi\, d\xi,\end{aligned} \tag{C 2}$$

and

$$\begin{aligned}\mathscr{L}_{1111} = {} & \iint \langle \phi_{11}\zeta_{11\theta}-(1+\lambda)^{-1}[\zeta_{11}^2+\lambda(\nabla\zeta_{11})^2-\tfrac{1}{4}\lambda(\nabla\zeta_1)^4]+\phi_{11\xi}\zeta_1\zeta_{1\theta} \\ & +\phi_{1\xi}(\zeta_1\zeta_{11})_\theta+\tfrac{1}{2}\phi_{1\xi\xi}\zeta_1^2\zeta_{1\theta}+\phi_1[-\phi_{1\xi\xi}\zeta_{11}-\phi_{11\xi\xi}\zeta_1-\tfrac{1}{2}\phi_{1\xi\xi\xi}\zeta_1^2 \\ & +\zeta_1\nabla\phi_{1\xi}\cdot\nabla\zeta_1+\nabla\phi_1\cdot\nabla\zeta_{11}+\nabla\phi_{11}\cdot\nabla\zeta_1]\rangle\,\rho\, d\rho\, d\psi,\end{aligned} \tag{C 3}$$

where the integrands in the free-surface and wavemaker integrals are projected onto $\xi = 0$ and $\rho = \rho_1$, respectively, and the corresponding limits of integration for ρ and ξ are (ρ_1, ∞) and $(-\infty, 0)$. The contributions of the end points $r-r_1 = \chi$ and $z = \zeta$ in (2.3) to this projection cancel in the present approximation (as in I).

Integrating (C 1) by parts with the aid of the identities

$$\frac{d}{d\rho} = \frac{\partial}{\partial\rho}+2\epsilon\frac{\partial}{\partial R}, \quad \phi_1\phi_{1\rho R} = \phi_{1\rho}\phi_{1R} \tag{C 4a, b}$$

and similarly for ζ_1, and

$$gk\iint[\zeta_1^2+\lambda(\nabla\zeta_1)^2]\,\rho\,\mathrm{d}\rho\,\mathrm{d}\psi = \omega_k^2\iint\zeta_1\,\Lambda\zeta_1\,\rho\,\mathrm{d}\rho\,\mathrm{d}\psi, \tag{C 4c}$$

and invoking $\langle\zeta_1\Lambda\zeta_1\rangle = -\langle\zeta_1\phi_{1\theta}\rangle = \langle\phi_1\zeta_{1\theta}\rangle$ and (4.3) for β, we obtain

$$\mathscr{L}_{11} = -2\iiint\langle\phi_{1R}^2\rangle\,\rho\,\mathrm{d}R\,\mathrm{d}\psi\,\mathrm{d}\xi-\left(\frac{2\lambda}{1+\lambda}\right)\iint\langle\zeta_{1R}^2\rangle\,\rho\,\mathrm{d}R\,\mathrm{d}\psi$$
$$-\iint\langle\phi_1\zeta_{1\tau}+\beta\phi_1\zeta_{1\theta}\rangle\,\rho\,\mathrm{d}R\,\mathrm{d}\psi, \tag{C 5}$$

which reduces to (4.2) through the substitution of ϕ_1 from (3.10).

Turning to $\mathscr{L}_{011}$, we consider first the contributions of ϕ_{11} and ζ_{11} in (C 2). It follows from Green's theorem, (3.6a), (3.8a) and (3.9) that

$$(1+\lambda)^{-1}\iint\langle\zeta_0\zeta_{11}+\lambda\nabla\zeta_0\cdot\nabla\zeta_{11}\rangle\,\mathrm{d}S = \iint\langle\zeta_{11}\Lambda\zeta_0\rangle\,\mathrm{d}S, \tag{C 6}$$

where, here and subsequently, $\mathrm{d}S \equiv \rho\,\mathrm{d}\rho\,\mathrm{d}\psi$. We transform the integral of $\langle-\phi_{11}\hat{\chi}_\theta\rangle$ according to

$$-\iint\langle\phi_{11}\hat{\chi}_\theta\rangle_{\rho=\rho_1}\rho_1\,\mathrm{d}\psi\,\mathrm{d}\xi = \iint\langle\phi_0\phi_{11\rho}-\phi_{11}\phi_{0\rho}\rangle_{\rho=\rho_1}\rho_1\,\mathrm{d}\psi\,\mathrm{d}\xi \tag{C 7a}$$

$$= \iint\langle\phi_0\phi_{11\xi}-\phi_{11}\phi_{0\xi}\rangle_{\xi=0}\,\mathrm{d}S, \tag{C 7b}$$

where (C 7a) follows from $\phi_{11\rho} = 0$ (3.8b) and $\phi_{0\rho} = \hat{\chi}_\theta$(3.6$b$) on $\rho = \rho_1$, (C 7b) follows from (C 7a) through Green's theorem and the hypothesis that

$$I_\infty \equiv \lim_{\rho\uparrow\infty}\iint\langle\phi_0\phi_{11\rho}-\phi_{11}\phi_{0\rho}\rangle\,\rho\,\mathrm{d}\rho\,\mathrm{d}\psi = 0, \tag{C 8}$$

and (C 8) follows from the radiation conditions satisfied by ϕ_0 and ϕ_{11} (see (A 11) and (B 13)). Combining (C 6) and (C 7b) with the first two terms in the integrand of (C 2) and invoking (3.5a, b), (3.7a) and the identity $\langle ab_\theta\rangle = -\langle a_\theta b\rangle$, we obtain

$$\langle\phi_{11}\zeta_{0\theta}+\phi_0\zeta_{11\theta}-2\zeta_{11}\Lambda\zeta_0+\phi_0\phi_{11\xi}-\phi_{11}\phi_{0\xi}\rangle$$
$$= \langle\phi_{11}(\zeta_{0\theta}-\phi_{0\xi})-2\zeta_{11}(\phi_{0\theta}+\Lambda\zeta_0)+\phi_0(\phi_{11\xi}-\zeta_{11\theta})\rangle = \langle\phi_0(-\phi_{1\xi\xi}\zeta_1+\nabla\phi_1\cdot\nabla\zeta_1)\rangle, \tag{C 9}$$

the substitution of which into (C 2) yields

$$\mathscr{L}_{011} = \iint\langle-\phi_{1\xi\xi}(\phi_0\zeta_1+\phi_1\zeta_0)-\phi_{0\xi\xi}\phi_1\zeta_1+\phi_{0\xi}\zeta_1\zeta_{1\theta}+\phi_1(\nabla\phi_0\cdot\nabla\zeta_1+\nabla\phi_1\cdot\nabla\zeta_0)$$
$$+\phi_{1\xi}(\zeta_0\zeta_1)_\theta+\phi_0\nabla\phi_1\cdot\nabla\zeta_1\rangle\,\mathrm{d}S+\iint\langle\phi_1(\phi_{1\rho\rho}\hat{\chi}-\phi_{1\xi}\hat{\chi}_\xi)\rangle\,\rho_1\,\mathrm{d}\psi\,\mathrm{d}\xi \tag{C 10a}$$

$$= \iint\langle\phi_1^2[\partial_\xi-\tfrac{1}{2}(1+\Lambda_\xi^{-1})\,\partial_\xi^2]\,\phi_{0\theta}\rangle\,\mathrm{d}S$$
$$+\iint\langle\phi_1(\phi_{1\rho\rho}+2\phi_1)\,\hat{\chi}\,\rho_1\,\mathrm{d}\psi\,\mathrm{d}\xi-\int\langle\phi_1^2[(1+2\Lambda_\xi^{-1})\,\hat{\chi}]_{\xi=0}\rangle\,\rho_1\,\mathrm{d}\psi, \tag{C 10b}$$

where (C 10b) follows from (C 10a) with the aid of Green's theorem and (3.4)–(3.6), and Λ_ξ is defined by (4.6). Substituting ϕ_1 from (3.10) into (C 10b) and invoking (A 1) and (3.6c), we obtain (4.5a).

The reduction of $\mathscr{L}_{1111}$ (C 3) is rather lengthy and involves repeated applications of Green's theorem, integration by parts with respect to θ, and simplification with the aid of (3.4) for ϕ_1 and ϕ_{11}, (3.5a, b) and (3.6a, b) for ϕ_1 and ζ_1, and (3.7a, b) and (3.8a, b) for ϕ_{11} and ζ_{11} to obtain

$$\mathscr{L}_{1111} = \iint \langle (-\phi_1\zeta_1 + \nabla\phi_1\cdot\nabla\zeta_1)\phi_{11} + [\zeta_1^2 - \tfrac{1}{2}(\nabla\phi_1)^2]\zeta_{11} - \phi_1^2\zeta_1^2 + 2\phi_1\zeta_1\nabla\phi_1\cdot\nabla\zeta_1 + \frac{1}{4}\left(\frac{\lambda}{1+\lambda}\right)(\nabla\zeta_1)^4\rangle\,\mathrm{d}S. \quad (C\ 11)$$

We remark that the coefficients of ϕ_{11} and ζ_{11} in (C 11) are equal, respectively, to the right-hand sides of (B 1a) and (B 1b). Substituting ϕ_1 and ζ_1 from (3.10) and ϕ_{11} and ζ_{11} from (B 3) into (C 11), we obtain

$$\mathscr{L}_{1111} = \tfrac{1}{2}\,\mathrm{Re}\iint |A|^4\left\{\mathscr{G}\Phi + \mathscr{H}Z + \tfrac{1}{2}\mathscr{G}\Lambda^{-1}\mathscr{G} - \mathscr{G}^2 + \left[1 + \frac{3}{4}\left(\frac{\lambda}{1+\lambda}\right)\right](\nabla\mathscr{F}_m)^4\right\}\mathrm{d}S, \quad (C\ 12)$$

where $\mathscr{G}$ and $\mathscr{H}$ are given by (B 2) and Φ and Z are given by (B 12).

Neglecting the oscillatory component of the integrand for $R = O(1)$, as in the approximation of $\mathscr{L}_{11}$ by (4.10), and approximating A by A_1 for $\rho = O(\rho_1)$, we obtain the asymptotic approximation

$$\mathscr{L}_{1111} = \tfrac{1}{2}Q\int_{R_1}^{\infty} |A|^4\frac{\mathrm{d}R}{R} + \tfrac{1}{2}q_{\mathrm{r}}|A_1|^4 + O(\epsilon), \quad (C\ 13)$$

where

$$Q = \frac{3}{8\pi}\left[1 + \left(\frac{1+\lambda}{1+4\lambda}\right) + \frac{9}{4}\left(\frac{\lambda}{1+\lambda}\right)\right] \quad (C\ 14a)$$

$$q_{\mathrm{r}} = \mathrm{Re}\iint\left\{\mathscr{G}\Phi + \mathscr{H}Z + \tfrac{1}{2}\mathscr{G}\Lambda^{-1}\mathscr{G} - \mathscr{G}^2 + \left[1 + \frac{3}{4}\left(\frac{\lambda}{1+\lambda}\right)\right](\nabla\mathscr{F}_m)^4 - \frac{4Q}{3\pi}\frac{\cos^4 m\psi}{\rho^2}\right\}\mathrm{d}S, \quad (C\ 14b)$$

and we have assumed that $\mu_0 \neq 2$. If $\mu_0 = 2$ the radiated components of Φ and Z, as approximated by (B 13), resonate with $\mathscr{G}$ and $\mathscr{H}$ (which reflects a Wilton (1915) resonance between the cross-wave and its second harmonic), and the integrand in (C 12) comprises a non-oscillatory (in ρ) component that is $O(\rho^{-\frac{3}{2}})$, in consequence of which the integral diverges. This resonance occurs for $\lambda = \frac{1}{2}$ and required a re-scaling for which ϕ_1 and ϕ_{11} have the same order of magnitude.

Appendix D. Radiation damping

It follows from (B 13) that the disturbance described by (B 1) radiates energy at $O(A^4)$, and it can be shown (cf. Miles 1990) that the corresponding radiation damping may be incorporated in the evolution equations by allowing q to be complex in (5.3a), with the imaginary part (cf. (C 14b))

$$q_{\mathrm{i}} = \mathrm{Im}\iint(\mathscr{G}\Phi + \mathscr{H}Z)\,\mathrm{d}S, \quad (D\ 1)$$

where $(\mathscr{G}, \mathscr{H})$ and (Φ, Z) are given by (B 2) and (B 12).

The imaginary part of $\mathscr{G}\Phi+\mathscr{H}Z$ is derived entirely from the indentation of the path of integration in (B 12) under the pole at $\mu=\mu_0$ and is given by

$$\operatorname{Im}(\mathscr{G}\Phi+\mathscr{H}Z)=\tfrac{1}{4}\pi(\mathscr{G}(\mu_0)-\tfrac{1}{2}\mu_0\,\mathscr{H}(\mu_0)]\,\mu_0(\mu_0^2+4\mu_0-8)(1+\lambda)(1+3\lambda\mu_0)^{-1}\sum_n \mathscr{K}_n(\mu_0)\,F_n(\mu_0\rho,\mu_0\rho_1)\cos n\psi, \tag{D 2}$$

where n is summed over 0 and $2m$. Combining (B 2) and (D 2) in (D 1) and carrying out the integration over S ($\mathrm{d}S=\rho\,\mathrm{d}\rho\,\mathrm{d}\psi$) with the aid of (B 9) and (B 11), we obtain

$$q_1=\tfrac{1}{2}\pi^2\mu_0^2(\tfrac{1}{4}\mu_0^2+\mu_0-2)^2(1+\lambda)(1+3\lambda\mu_0)^{-1}[\mathscr{K}_0^2(\mu_0)+\tfrac{1}{2}\mathscr{K}_{2m}^2(\mu_0)]. \tag{D 3}$$

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